

4. The matrix  $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$  obviously has linearly dependent columns (because one column is zero), and so the matrix is not invertible (or singular) by (e) in the IMT.

$$5. \begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is not invertible because it is not row equivalent to the identity matrix.

$$6. \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is not invertible because it is not row equivalent to the identity matrix.

$$7. \begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The  $4 \times 4$  matrix has four pivot positions and so is invertible by (c) of the IMT.

8. The  $4 \times 4$  matrix  $\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$  is invertible because it has four pivot positions, by (c) of the IMT.

$$9. [M] \begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ 4 & 0 & -7 & -7 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & -11 & -7 & 15 \\ 0 & 9 & 31 & 12 \\ 0 & 8 & 5 & -11 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 9 & 31 & 12 \\ 0 & -11 & -7 & 15 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 25.375 & 24.375 \\ 0 & 0 & -1.250 & -1.250 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 25.375 & 24.375 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 25.375 & 24.375 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The  $4 \times 4$  matrix is invertible because it has four pivot positions, by (c) of the IMT.

Also, you could ask students to explain why an  $n \times n$  matrix with linearly independent columns must also have linearly independent rows.

15. If  $A$  has two identical columns then its columns are linearly dependent. Part (e) of the IMT shows that  $A$  cannot be invertible.
16. Part (h) of the IMT shows that a  $5 \times 5$  matrix cannot be invertible when its columns do not span  $\mathbf{R}^5$ .
17. If  $A$  is invertible, so is  $A^{-1}$ , by Theorem 6 in Section 2.2. By (e) of the IMT applied to  $A^{-1}$ , the columns of  $A^{-1}$  are linearly independent.
18. By (g) of the IMT,  $C$  is invertible. Hence, each equation  $C\mathbf{x} = \mathbf{v}$  has a unique solution, by Theorem 5 in Section 2.2. This fact was pointed out in the paragraph following the proof of the IMT.
19. By (e) of the IMT,  $D$  is invertible. Thus the equation  $D\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$  in  $\mathbf{R}^7$ , by (g) of the IMT. Even better, the equation  $D\mathbf{x} = \mathbf{b}$  has a *unique* solution for each  $\mathbf{b}$  in  $\mathbf{R}^7$ , by Theorem 5 in Section 2.2. (See the paragraph following the proof of the IMT.)
20. By the box following the IMT,  $E$  and  $F$  are invertible and are inverses. So  $FE = I = EF$ , and so  $E$  and  $F$  commute.
21. The matrix  $G$  cannot be invertible, by Theorem 5 in Section 2.2 or by the box following the IMT. So (h) of the IMT is false and the columns of  $G$  do not span  $\mathbf{R}^n$ .
22. Statement (g) of the IMT is false for  $H$ , so statement (d) is false, too. That is, the equation  $H\mathbf{x} = \mathbf{0}$  has a nontrivial solution.
23. Statement (b) of the IMT is false for  $K$ , so statements (e) and (h) are also false. That is, the columns of  $K$  are linearly *dependent* and the columns do *not* span  $\mathbf{R}^n$ .
24. No conclusion about the columns of  $L$  may be drawn, because no information about  $L$  has been given. The equation  $L\mathbf{x} = \mathbf{0}$  *always* has the trivial solution.
25. Suppose that  $A$  is square and  $AB = I$ . Then  $A$  is invertible, by the (k) of the IMT. Left-multiplying each side of the equation  $AB = I$  by  $A^{-1}$ , one has
 
$$A^{-1}AB = A^{-1}I, \quad IB = A^{-1}, \quad \text{and} \quad B = A^{-1}.$$
 By Theorem 6 in Section 2.2, the matrix  $B$  (which is  $A^{-1}$ ) is invertible, and its inverse is  $(A^{-1})^{-1}$ , which is  $A$ .
26. If the columns of  $A$  are linearly independent, then since  $A$  is square,  $A$  is invertible, by the IMT. So  $A^2$ , which is the product of invertible matrices, is invertible. By the IMT, the columns of  $A^2$  span  $\mathbf{R}^n$ .
27. Let  $W$  be the inverse of  $AB$ . Then  $ABW = I$  and  $A(BW) = I$ . Since  $A$  is square,  $A$  is invertible, by (k) of the IMT.

**Note:** The *Study Guide* for Exercise 27 emphasizes here that the equation  $A(BW) = I$ , *by itself*, does not show that  $A$  is invertible. Students are referred to Exercise 38 in Section 2.2 for a counterexample. Although there is an overall assumption that matrices in this section are square, I insist that my students mention this fact when using the IMT. Even so, at the end of the course, I still sometimes find a student who thinks that an equation  $AB = I$  implies that  $A$  is invertible.

28. Let  $W$  be the inverse of  $AB$ . Then  $WAB = I$  and  $(WA)B = I$ . By (j) of the IMT applied to  $B$  in place of  $A$ , the matrix  $B$  is invertible.

29. Since the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is not one-to-one, statement (f) of the IMT is false. Then (i) is also false and the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  does not map  $\mathbf{R}^n$  onto  $\mathbf{R}^n$ . Also,  $A$  is not invertible, which implies that the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is not invertible, by Theorem 9.
30. Since the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one, statement (f) of the IMT is true. Then (i) is also true and the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbf{R}^n$  onto  $\mathbf{R}^n$ . Also,  $A$  is invertible, which implies that the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is invertible, by Theorem 9.
31. Since the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$ , the matrix  $A$  has a pivot in each row (Theorem 4 in Section 1.4). Since  $A$  is square,  $A$  has a pivot in each column, and so there are no free variables in the equation  $A\mathbf{x} = \mathbf{b}$ , which shows that the solution is unique.

**Note:** The preceding argument shows that the (square) shape of  $A$  plays a crucial role. A less revealing proof is to use the “pivot in each row” and the IMT to conclude that  $A$  is invertible. Then Theorem 5 in Section 2.2 shows that the solution of  $A\mathbf{x} = \mathbf{b}$  is unique.

32. If  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution, then  $A$  must have a pivot in each of its  $n$  columns. Since  $A$  is square (and this is the key point), there must be a pivot in each row of  $A$ . By Theorem 4 in Section 1.4, the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$  in  $\mathbf{R}^n$ .

Another argument: Statement (d) of the IMT is true, so  $A$  is invertible. By Theorem 5 in Section 2.2, the equation  $A\mathbf{x} = \mathbf{b}$  has a (unique) solution for each  $\mathbf{b}$  in  $\mathbf{R}^n$ .

33. (Solution in *Study Guide*) The standard matrix of  $T$  is  $A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$ , which is invertible because  $\det A \neq 0$ . By Theorem 9, the transformation  $T$  is invertible and the standard matrix of  $T^{-1}$  is  $A^{-1}$ . From the formula for a  $2 \times 2$  inverse,  $A^{-1} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$ . So

$$T^{-1}(x_1, x_2) = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (7x_1 + 9x_2, 4x_1 + 5x_2)$$

34. The standard matrix of  $T$  is  $A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$ , which is invertible because  $\det A = 2 \neq 0$ . By Theorem 9,

$T$  is invertible, and  $T^{-1}(\mathbf{x}) = B\mathbf{x}$ , where  $B = A^{-1} = \frac{1}{2} \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$ . Thus

$$T^{-1}(x_1, x_2) = \frac{1}{2} \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left( \frac{7}{2}x_1 + 4x_2, \frac{5}{2}x_1 + 3x_2 \right)$$

35. (Solution in *Study Guide*) To show that  $T$  is one-to-one, suppose that  $T(\mathbf{u}) = T(\mathbf{v})$  for some vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbf{R}^n$ . Then  $S(T(\mathbf{u})) = S(T(\mathbf{v}))$ , where  $S$  is the inverse of  $T$ . By Equation (1),  $\mathbf{u} = S(T(\mathbf{u}))$  and  $S(T(\mathbf{v})) = \mathbf{v}$ , so  $\mathbf{u} = \mathbf{v}$ . Thus  $T$  is one-to-one. To show that  $T$  is onto, suppose  $\mathbf{y}$  represents an arbitrary vector in  $\mathbf{R}^n$  and define  $\mathbf{x} = S(\mathbf{y})$ . Then, using Equation (2),  $T(\mathbf{x}) = T(S(\mathbf{y})) = \mathbf{y}$ , which shows that  $T$  maps  $\mathbf{R}^n$  onto  $\mathbf{R}^n$ .

Second proof: By Theorem 9, the standard matrix  $A$  of  $T$  is invertible. By the IMT, the columns of  $A$  are linearly independent and span  $\mathbf{R}^n$ . By Theorem 12 in Section 1.9,  $T$  is one-to-one and maps  $\mathbf{R}^n$  onto  $\mathbf{R}^n$ .

36. If  $T$  maps  $\mathbf{R}^n$  onto  $\mathbf{R}^n$ , then the columns of its standard matrix  $A$  span  $\mathbf{R}^n$ , by Theorem 12 in Section 1.9. By the IMT,  $A$  is invertible. Hence, by Theorem 9 in Section 2.3,  $T$  is invertible, and  $A^{-1}$  is the standard matrix of  $T^{-1}$ . Since  $A^{-1}$  is also invertible, by the IMT, its columns are linearly independent and span  $\mathbf{R}^n$ . Applying Theorem 12 in Section 1.9 to the transformation  $T^{-1}$ , we conclude that  $T^{-1}$  is a one-to-one mapping of  $\mathbf{R}^n$  onto  $\mathbf{R}^n$ .